An Introduction to Derivatives and Risk Management, 10th Edition

Don M. Chance and Robert Brooks

Technical Note: The Arbitrage Principle

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This technical note explains how arbitrage is eliminated in a well-functioning financial market. It uses the simple case of an asset with two unknown outcomes and a risk-free bond.

Suppose the asset is currently priced at S, and one period later it can go up to S^+ or S^-. While there are probabilities that define how likely each outcome is, we do not know or care what they are. We also allow the existence of a risk-free bond that will pay an interest rate of r. Thus, \$1 invested in this bond will pay off $1 + r$ one period later regardless of which way the asset moves.

Let us consider two possible cases. Suppose $S^-/S > 1 + r$ and $1 + r > S^+/S$. In the first case, the S^-/S is 1 plus the return on the stock if the worse of the two outcomes occurs. In the second case, S^+/S is 1 plus the return on the stock if the better of the two outcomes occurs, with "worse" and "better" meaning the "lower" and "higher" stock prices. In the first case, $S^-/S > 1 + r$, then the worst case outcome for the stock is better than the return on the risk-free asset. In that case, we could borrow S dollars at the risk-free rate and be assured that we would earn more from the stock than would be necessary to pay off the loan. In that case, everyone would do it so the stock price would rise from all of the buying until $S^-/S < 1 + r$. In the second case, we could sell short the stock and use the money to buy the risk-free bond and be assured that we would have more than enough money from the interest on the bond to cover the worst possible outcome from shorting the stock and having to buy it back at the higher of the two possible prices. Naturally investors would jump at this opportunity and the stock price would fall from all of the shorting until $1 + r > S^+/S$.

Thus, we have two conditions that have to hold to avoid an arbitrage opportunity that would otherwise be quickly exploited:

$$S^-/S < 1 + r$$

$$1 + r > S^+/S$$

This means that we have to have

$$S^-/S < 1 + r < S^+/S.$$

In other words, the risk-free rate must lie between the lower return on the stock and the upper return on the stock. Otherwise, there would be an arbitrage property.

Let's write the above expression as

$$a < b < c$$

where $a = S^-/S$, $b = 1 + r$, and $c = S^+/S$. If the above expression is true, a simple rule of mathematics states that it is possible to take a weighted average of a and c to obtain b. Let's define the weights as p and $1 - p$. Thus,

$$b = (1 - p)a + pc$$

or

$$(1 + r) = p(S^+/S) + (1 - p)S^-/S.$$

This means that

$$p = \frac{1 + r - (S^- / S)}{S^+ / S - S^- / S}.$$

The value p is called the *risk neutral* or equivalent martingale probability and plays a very important role in pricing options. But what we have seen here is that it is not necessary to introduce an option to establish that such a probability exists if a market allows no arbitrage opportunities.

In other words, when a financial market permits no arbitrage, the risky asset must not be dominated by the risk-free asset or vice versa. That is, it must not be possible to buy one asset and sell the other and be guaranteed to earn money without having to invest money. This is the essential idea behind the arbitrage rule.

References

The general theorem of no-arbitrage trading is found in:

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Jarrow, R. A. and S. M. Turnbull. *Derivative Securities*, 2nd. ed. Cincinnati: South-Western College Publishing (1999), Ch. 6.

Neftci, S. *An Introduction to the Mathematics of Financial Derivatives*, 2nd. ed. San Diego: Academic Press (2000), Ch. 3.